

On a very steep version of the standard map.

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September 26, 2016

We consider the long time behavior of the trajectories of the discontinuous analog of the standard Chirikov map. We prove that for some values of parameters all the trajectories remains bounded for all time. For other set of parameters we provide an estimate for the escape rate for the trajectories and present a numerically supported conjecture for the actual escape rate.

1 Introduction.

We consider the area-preserving transformation of the cylinder $[0, 1) \times \mathbb{R}$ defined by $f(x, y) = (x', y')$ where

$$\begin{cases} x' &= x + \alpha y \pmod{1} \\ y' &= y + \operatorname{sgn}\left(x' - \frac{1}{2}\right), \end{cases} \quad (1)$$

Parameter $\alpha \in \mathbb{R}$ is called the twist parameter. A point at position (x, y) on the cylinder moves at constant height y around the cylinder a distance αy , and then moves up one unit if it is on the right half of the cylinder ($x' \in (1/2, 1)$), down one unit if it is on the left half ($x' \in (0, 1/2)$), and stays at the same vertical position if it is at the singular lines $x' = 1/2$ or 0 .

Such system can be regarded as a discontinuous analog of the standard Chirikov map (see [3]), where the smooth function $\sin(x')$ is replaced by the discontinuous $\operatorname{sgn}(x')$. This system can be also obtained from the Fermi-Ulam accelerator model with the sawtooth-like wall movement regime (see [1] for details). For the smooth variants of the described problems KAM-technique can be used to provide the existence of the

invariant curves separating the phase space and so no unbounded orbit exists for such systems. Since transformation (1) is discontinuous, KAM theory is not applicable and so new methods are needed for the analysis. Such systems having many interesting dynamical properties, attracted a lot of attention in the past few years (see e.g. [5], [4]).

In this note we study the asymptotic properties of the orbits of system (1) in terms of the growth rate of the height y_n of the iterates $(x_n, y_n) = f^n(x_0, y_0)$. We will focus on the rational values of the twist parameter α . The case of the irrational values of α is more difficult and will be a subject of a future work.

In the next section, we collect preliminary results on the structure of the set of orbits of system (1) and relate our system to a transformation on a finite lattice. In section 3 we present our main results and state some conjectures based on the numerical simulations. Section 4 is devoted to the numerical study of the periodic orbits.

Acknowledgments. Present work was done during the Summer@ICERM research program in 2015. Authors are deeply thankful to ICERM and Brown University for the hospitality and highly encouraging atmosphere. Authors also want to thank Vadim Zharnitsky and Stefan Klajbor-Goderich for deep and fruitful discussions.

2 Preliminaries.

We will use the following notations. Integer part of x is denoted as $\lfloor x \rfloor$, therefore for the fractional part of x we have $\{x\} = x - \lfloor x \rfloor$. $\mathbb{Z}_q = \{0, 1, \dots, q-1\}$ will denote the ring of residues modulo q .

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Lemma 1. Map (1) is symmetric with respect to the point $(1/2, 0)$. In details, for any two points (x, y) and (\tilde{x}, \tilde{y}) such that $(x, y) + (\tilde{x}, \tilde{y}) = (1, 0)$ one has $f(x, y) + f(\tilde{x}, \tilde{y}) = (1, 0)$ (See Fig. 1).

Proof. Let $(\tilde{x}, \tilde{y}) = (1 - x, -y)$, then $x' = x + \alpha y$ and $\tilde{x}' = 1 - x - \alpha y = 1 - x'$. Hence $\tilde{y}' - \tilde{y} = -(y' - y)$. \square

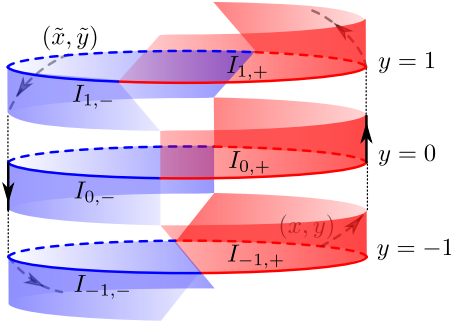


Figure 1: Transformation (1) can be thought as interval exchange transformation with infinitely many intervals. Intervals $I_{j,+}$ are the pre-images of the segments $x \in (1/2, 1)$, $y = y_0 + (j + 1)$. Similarly, $I_{j,-}$ are the pre-images of $x \in (0, 1/2)$, $y = y_0 + (j - 1)$.

Next we notice that the transformation (1) preserves the lattice $y \in \{y_0 + \mathbb{Z}\}$ in the second coordinate. Let $\alpha = \frac{p}{q}$, where $p, q \in \mathbb{Z}$ and $\gcd(p, q) = 1$. Then for two points (x, y) and $(x, y \pm q)$ first components of their images coincide and thus the increments in the second components are equal. Therefore we can restrict our attention to the set $(x, y) \in [0, 1) \times \{y_0 + \mathbb{Z}_q\}$. Transformation (1) can be regarded as an interval exchange transformation on the union of $2q$ intervals $\bigcup_{j=1}^q (I_{j,+} \cup I_{j,-})$, where $I_{j,-} = \{(x, y) : \{qx + py\} < q/2\}, y = y_0 + j\}$ and $I_{j,+} = \{(x, y) : \{qx + py\} > q/2\}, y = y_0 + j\}$ (see Fig. 1)

If one considers the special case $q = 1$ dynamics of the system (1) degenerates to

$$\begin{cases} x' = x + py_0 \pmod{1} \\ j' = j + \operatorname{sgn}(x' - \frac{1}{2}) \end{cases}$$

In this particular case dynamic of the first coordinate became independent from the second coordinate.

Therefore one can identify all the intervals $I_{j,+} = I_+$ and all the intervals $I_{j,-} = I_-$. For $py_0 = 1$ one immediately obtain linearly growing trajectory $f^n(1/4, y_0) = (1/4, n + y_0)$. On the other hand for $py_0 = 1/2$ any trajectory remains bounded since for $x \in I_+$ from lemma 1 it follows that $x' \in I_-$. The case of y_0 being irrational has been extensively studied (see [9, 7, 8, 2]) It provides a random-like behavior of the trajectories depending on the arithmetic properties of the initial condition y_0 .

In this paper we address the case $q > 1$ and consider rational initial conditions $y_0 = \frac{a}{b}$. Using substitution $y = y_0 + j$, we rewrite transformation (1) as

$$\begin{cases} x' = x + \frac{p(a+bj)}{bq} \pmod{1} \\ j' = j + \operatorname{sgn}(x' - \frac{1}{2}) \end{cases} \quad (2)$$

where j' is defined by the expression $y' = y_0 + j'$.

Lemma 2. Trajectories of the system (2) are organized in bands: for (x, j) and (\tilde{x}, j) such that $\lfloor bqx \rfloor = \lfloor bq\tilde{x} \rfloor$ it follows that $\lfloor bqx' \rfloor = \lfloor bq\tilde{x}' \rfloor$.

Proof. Obviously, integral parts of xbq and $\tilde{x}bq$ are changed by the transformation (2) by the same amount $p(a + bj)$. \square

From lemma 2 it follows that we can restrict our attention on the single representatives from the classes of equivalent trajectories and consider our transformation on the discrete torus $(x, y) \in \mathbb{Z}_{bq} \times \mathbb{Z}_q$. For the sake of simplicity we will use the following lattices:

$$L_e = \left\{ \left(\frac{2 + 4r}{4bq}, \frac{a}{b} + j \right), j \in \mathbb{Z}_q, r \in \mathbb{Z}_{bq} \right\}$$

$$L_o = \left\{ \left(\frac{3 + 4r}{4bq}, \frac{a}{b} + j \right), j \in \mathbb{Z}_q, r \in \mathbb{Z}_{bq} \right\}$$

We refer to L_e and L_o as the even and the odd lattice, respectively. L_o can be obtained from L_e by shifting to the right by $\frac{1}{4bq}$ (see Fig. 2).

Thanks to lemma 2 these lattices are invariant under the action of f . To simplify the notations, henceforth when bq is even we consider $f : L_e \rightarrow L_e$, and when

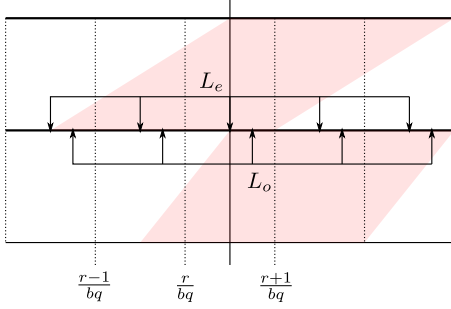


Figure 2: Lattices L_e and L_o for odd bq .

bq is odd we consider $f : L_o \rightarrow L_o$. For purposes of calculation we will think of f as acting on (r, j) instead of (x, y) . Explicitly, for $r \in \mathbb{Z}_{bq}$ and $j \in \mathbb{Z}_q$ we have

$$\begin{cases} r' = r + p(a + bj) & (\text{mod } bq) \\ j' = j + \text{sgn}(2r' - bq + 1 + \delta) & (\text{mod } q), \end{cases} \quad (3)$$

where $\delta = bq \pmod{2}$ refers to our choice of the lattice L_e or L_o .

3 Main Results.

Since the lattices L_e, L_o are finite all the trajectories of the system (3) are periodic. The total increment in the second coordinate of any periodic trajectory has to be proportional to q . If the total increment of a trajectory is zero, we will call such a trajectory bounded or periodic. Otherwise the trajectory will be called escaping.

3.1 Existence of escaping trajectories.

Theorem 1. Let $\alpha = \frac{p}{q}$ and $y_0 = \frac{a}{b}$ be two rational numbers satisfying the condition $\lfloor bq/2 \rfloor = pa \pmod{b}$. Then

1. For bq even, any orbit of the transformation (1) starting at the level $y = y_0$ is bounded.
2. For bq odd there exists a unique class of equivalent trajectories of the system (1) starting at the level $y = y_0$ and growing without bounds.

Proof. Thanks to the above discussion, every unbounded trajectory of the system (1) corresponds to the escaping trajectory of the system (3). Thus, one has to show that system (3) either does not have escaping trajectories (for even bq) or has exactly one escaping trajectory (for odd bq). Transformation (3) can be considered as a continuous transformation with respect to the second coordinate, since every iteration gets an increment of ± 1 in j . We will construct a critical level $j = j_*$ such that no trajectory can cross it in the case of even bq . As it will be clear from the construction, for the case of odd bq there is only one trajectory which can cross this level. We will look for j_* such that

$$p(a + bj_*) = \lfloor bq/2 \rfloor \pmod{bq}. \quad (4)$$

Note that p/q is irreducible and so $\gcd(pb, bq) = b$. Since by the assumption of the theorem $\lfloor bq/2 \rfloor - pa$ is divisible by b we conclude that the congruence (4) has exactly b solutions in the form $j_* + kq$, $k = 0, 1, \dots, (b-1)$ (see [6]). Thus all these solutions correspond to the same equivalence class in \mathbb{Z}_q .

We will show that in the case of even bq no trajectories may cross the level $j = j_*$ and for the odd q there is only one such trajectory. Indeed, assume for definiteness that for $(r, j_* - 1)$ we have $f(r, j_* - 1) = (r', j_*)$. This means that r' belongs to the right half of the cylinder. But for the even bq it follows that $bpj_* = bq/2$ and so r' is shifted exactly by $bq/2$ thus the trajectory coming to the critical level from below will go down at the next step. From lemma 1 it follows that neither trajectory can cross this level from above.

For the case of odd bq one gets $pj_* = (bq-1)/2$ and so only $r = (bq-1)/2$ together with its image $r' = bq-1$ belong to the right half of the cylinder. Since there is a unique point at which trajectory may pass the level j_* such a trajectory necessarily has to be escaping (see Fig. 5) \square

Remark 1. In the case of integral initial condition $y_0 = a$ one can set $b = 1$ and so the congruence (4) always has a solution. Thus theorem 1 states that for $\alpha = 1/2k$ all the trajectories of the system (1) remain bounded while for $\alpha = 1/(2k+1)$ there is only one equivalence class of unbounded trajectories.

From numerical simulations the following statement is evident.

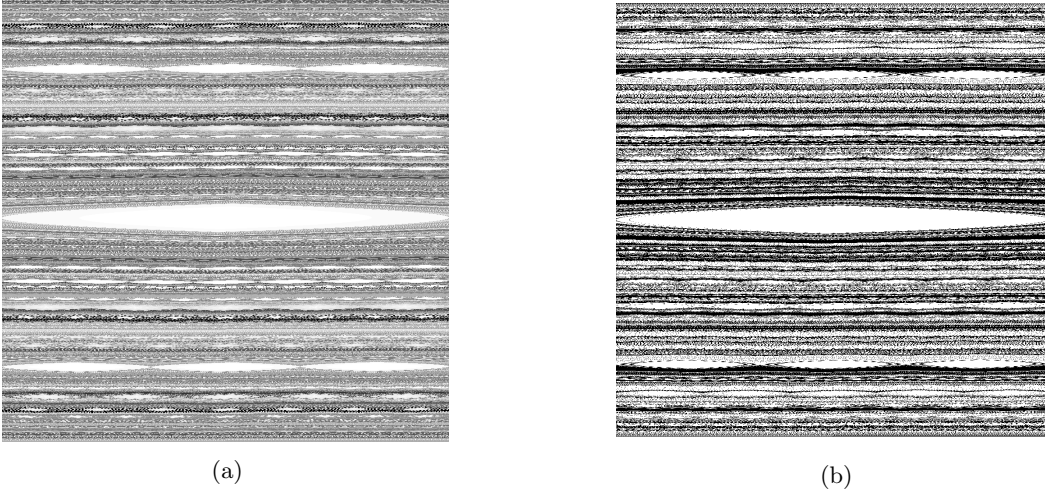


Figure 3: (a): Phase space for $q = 992$ is filled with the periodic orbits. Orbits are colored with the length of the period. Lighter points correspond to the shorter periods. Largest period equals 6168. (b): Escaping orbit for $q = 991$. Length of the orbit equals to 414639.

Conjecture 1. *For every $\alpha = p/q$, there exists $y_0 = a/b$ such that there is an escaping orbit of (1) starting at the level $y = y_0$.*

For odd q it follows from theorem 1 that $a = 0, b = 1$ provides the desired result. When the technical conditions of the Theorem 1 is not satisfied, congruence (4) has no solutions and we cannot construct the bottle-neck level passing which will assure that trajectory is escaping. However, one particular case seems to be tractable. From here on we fix p to be equal to 1.

Theorem 2. *For $q = 4k + 2$ there exists an escaping trajectory of the transformation f , starting at the level $y_0 = 1/2$.*

Letting $a = 1, b = 2$ we will show that the orbit of the point $(r_0, j_0) = (4k-1, 2k+1)$ is unbounded. From (3) we get the following system

$$\begin{cases} r' = r + 1 + 2j \pmod{2q} \\ j' = j + \text{sgn}(1 + 2r' - 2q) \pmod{q} \end{cases}$$

Next lemma provides us some control on the sub-lattice for which the orbit of (r_0, y_0) should belong to.

Lemma 3. *Let $q = 4k + 2$. Denote by (r_m, j_m) the m -th iterate of the point $(r_0, j_0) = (4k - 1, 2k + 1)$. Then $r_m + m \pmod{2} = 3 \pmod{4}$.*

Proof. At first we observe that the parity of the second coordinate of the point always differs from the parity of m . Indeed j_0 is odd and at each step the trajectory gets or loses 1.

We proceed by induction. First let us consider $m = 0$. We have $j_0 = 2k + 1 = 1 \pmod{2}$ and $r_0 = 4k - 1 = 3 \pmod{4}$. Then for $m = 1$ we get $r_1 = r_0 + 2j_0 + 1 = 3 + 2 + 1 = 2 \pmod{4}$.

Now assume that for some even m the assumption of the lemma holds true. Then since $j_m = 1 \pmod{2}$ it follows $r_{m+1} = r_m + 2j_m + 1 = 2 \pmod{4}$. Finally if the assumption holds for some odd m we get $j_m = 0 \pmod{2}$ and therefore $r_{m+1} = 2 + 1 + 4 \pmod{4} = 3 \pmod{4}$. \square

Proof of theorem 2. Consider the orbit of the point $(r_0, j_0) = (4k - 1, 2k + 1)$. One can easily calculate that $j_1 = j_0 + 1$ and $j_2 = j_1 + 1$, that is, there are immediately two consecutive increases. It then suffices to show that there is no point at level $j = 2k + 3$ from which there are two consecutive decreases.

To have a decrease to the $j = 2k + 1$ level from $(r, 2k + 2)$ we need to have $1 + 2r' - 2q < 0$, i.e.,

$$r + q + 3 \pmod{2q} < q - \frac{1}{2} \quad (5)$$

For $r < q - 3$ we have $r + q + 3 \pmod{2q} = r + q + 3$, so for the inequality (5) to hold we need $r < -\frac{1}{2} - 3$, which is impossible. For $r \geq q - 3$ we have $r + q + 3 \pmod{2q} = r - q + 3$, so for (5) to hold we need $r < 2q - \frac{7}{2}$. By lemma 3, $r = 2 \pmod{4}$ for any point in our desired orbit at level $j = 2k + 2$, and so the only r 's that are possibly in the orbit and result in a decrease from this level are $r = 4k + 2, 4k + 6, \dots, 8k - 2$.

In the $j = 2k + 3$ level, to have a decrease to the $j = 2k + 2$ level we need to have $r' < 4k + \frac{3}{2}$. To have two consecutive decreases, this r' must be one of the r 's we found in the previous paragraph. But the smallest such r is $4k + 2$, so this cannot happen. \square

For $q = 4k$, we searched for $y_0 = a/b$ that give an escaping orbit for some $(x_0, y_0) \in L_e$. We present the table of a/b depending on k with the smallest b .

k	1	2	3	4	5	6	7	8
a	1	4	4	1	26	36	67	63
b	3	13	11	45	57	103	144	205
k	\dots	9	10	11	12	13	14	
a	\dots	77	19	23	360	243	23	
b	\dots	227	337	223	1043	1264	505	

One can see that the b required increases rather quickly with k . It also appears that one cannot simply narrow the search by taking $a = 1$. For example, for $k = 3$ we searched for escaping orbits with $y_0 = 1/b$ and found none for $b \leq 5000$.

3.2 Length of the escaping orbit.

Now we will investigate the growth rate of the escaping orbit. The fastest possible rate for the transformation (1) is linear, i.e. the trajectory may gain as much as $O(N)$ in the second coordinate after N iterations. In fact, escaping trajectories grow much slower. Since the phase space of the transformation (3) is finite and thus so are all the trajectories we will consider the lengths of the trajectories instead of their growth rates. Let $\alpha =$

$1/q$ and $y_0 = 0$. Theorem 1 provides unique escaping trajectory for each odd q .

Definition 1. Define $\ell(q)$ as the unique odd-length period under f on L_o .

Quantity $\ell(q)$ describes the portion of the phase space $\mathbb{Z}_q \times \mathbb{Z}_q$ swiped by the escaping trajectory. Thus linearly growing trajectory should have $\ell(q) = O(q)$, since such a trajectory should visit every level of the lattice $O(1)$ amount of times. In fact, numerical experiments show that the escaping trajectory has the slowest possible growth rate $\ell(q) = O(q^2)$ (see Fig. 4)

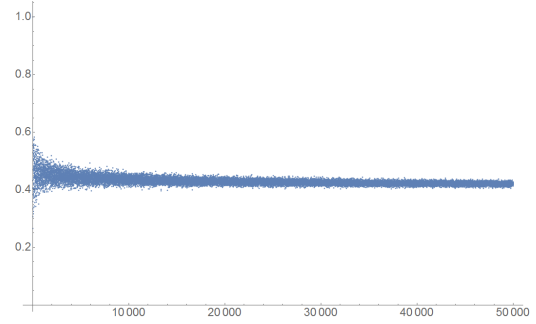


Figure 4: $\ell(q)/q^2$ for $q < 50\,000$. The average value of $\ell(q)/q^2$ was found to be about 0.43.

Conjecture 2. Length of the escaping orbit $\ell(q)$ grows as $O(q^2)$.

At this moment we can provide much milder estimate.

Theorem 3. Consider transformation (1) with $\alpha = \frac{1}{q}$ with odd q and $y_0 = 0$. Let $\{x_n, y_n\}$ denote the unbounded trajectory provided by theorem 1. Then for any n and n' such that $|n' - n| < q \log q$ it follows that $|y_{n'} - y_n| < q$.

We will prove this theorem providing an apriori bound on the length $\ell(q)$. The idea of the proof consists in the estimate of the time it takes from the escaping trajectory to pass the levels near the bottleneck level $j = (q + 1)/2$. We will use two lemmas. First lemma states that two consecutive vertical increases near $j = (q + 1)/2 + m$ (for $m \geq 0$ reasonably small)

cause the resulting iterate to be less than about k units to the right of $x = 1/2$ (a line of discontinuity for f). Second lemma uses the information about the first coordinate of the trajectory to estimate the time trajectory will spent on the prescribed level (see Fig 5).

Lemma 4. *Let $q \geq 1$ be odd. Suppose $(r_0, j_0) \in L_o$, $(r_1, j_1) = f(r_0, j_0)$, and $(r_2, j_2) = f(r_1, j_1)$ are such that $j_0 = \frac{q+1}{2} + m - 2$, $j_1 = j_0 + 1$, and $j_2 = j_0 + 2$ for some $m \in \{1, 2, \dots, \frac{q-1}{2}\}$. Then $r_2 \in [\frac{q-1}{2}, \frac{q-1}{2} + m - 1]$.*

Proof. Let j_0, j_1 , and j_2 be as in the statement of the lemma. That $j_1 > j_0$ means $r_1 > q/2$ (since we need $\text{sgn}(r_1 - \frac{q}{2}) = 1$ in order for this to happen), and in the same way $j_2 > j_1 \implies r_2 > q/2$. Therefore $\frac{q-1}{2} \leq r_i \leq q-1$ for $i \in \{1, 2\}$. These inequalities make sense, despite \mathbb{Z}_q not being ordered, because everything is in the interval $[0, q)$.

Each value of r_1 satisfying these inequalities can be written as $r_1 = \frac{q-1}{2} + n$ for some $n \in \{0, 1, \dots, \frac{q-1}{2}\}$. We have

$$\begin{aligned} r_2 &= r_1 + j_1 \pmod{q} = \left(\frac{q-1}{2} + n \right) + \\ &+ \left(\frac{q+1}{2} + m - 1 \right) \pmod{q} = n + m - 1 \end{aligned}$$

Using our definitions of n and m , we obtain

$$\frac{q-1}{2} \leq r_2 = n + m - 1 \leq \frac{q-1}{2} + m - 1$$

as desired. \square

The next lemma roughly states that if we start at a point within k units horizontally of $r = \frac{q+1}{2}$ (for $m > 0$ reasonably small) and at vertical level $j = \frac{q+1}{2} + m$, then the iterates of the point bounce at least about $q/2m$ times between $j = \frac{q+1}{2} + m$ and $j = \frac{q+1}{2} + m - 1$.

Lemma 5. *Let $q \geq 9$ be an odd integer. Let $j_0 = \frac{q+1}{2} + m$ and $\frac{q+1}{2} \leq r_0 \leq \frac{q+1}{2} + m - 1$ for $m \in \{1, 2, \dots, \lfloor q/9 \rfloor\}$. Let $(r_0, j_0) \in L_o$ and take N_m to be the greatest integer such that $j_n \in \{\frac{q+1}{2} + m, \frac{q+1}{2} + m - 1\}$ for all $n \leq N_m$. Then $N_m \geq \lfloor \frac{q-1}{2m} \rfloor - 1$.*

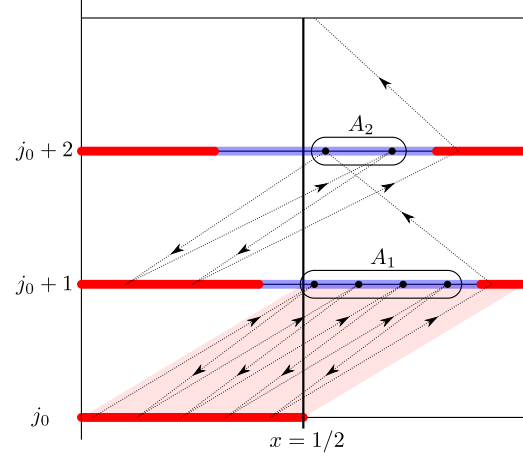


Figure 5: Escaping trajectory near the bottleneck level.

Proof. Let $r_0 = \frac{q+1}{2} + s$, where s is in $\{0, 1, \dots, m-1\}$. We have

$$r_1 = r_0 + j_0 \pmod{q} = s + m + 1 < \frac{q}{2},$$

$$j_1 = j_0 - 1 = \frac{q+1}{2} + m - 1$$

and

$$r_2 = r_1 + j_1 \pmod{q} = \frac{q+1}{2} + s + 2m > \frac{q}{2},$$

$$j_2 = j_1 + 1 = \frac{q+1}{2} + m.$$

Every two iterations of f , the value of r increases by the amount $(\frac{q+1}{2} + m) + (\frac{q+1}{2} + m - 1) \pmod{q} = 2m$ until r increases past q . Thus,

$$(r_{2n-1}, j_{2n-1}) = \left(s + (2n-1)m + 1, \frac{q+1}{2} + m - 1 \right)$$

and

$$(r_{2n}, j_{2n}) = \left(\frac{q+1}{2} + s + 2nm, \frac{q+1}{2} + m \right)$$

for all integers n with $0 \leq n \leq n^*$, where n^* is such that $s + (2n^* - 1)m + 1 < \frac{q}{2}$ and $\frac{q+1}{2} + s + 2mn^* < q$.

We claim that $n^* = \left\lfloor \frac{q-1}{4m} \right\rfloor - 1$ satisfies these inequalities. We have

$$s + (2n^* - 1)m + 1 < m - 1 + \left(\frac{q-1}{2m} - 1 \right) m + 1 = \frac{q-1}{2}$$

And on the other hand

$$\begin{aligned} \frac{q+1}{2} + s + 2mn^* &\leq \frac{q+1}{2} + m - 1 + \\ &+ 2 \left(\frac{q-1}{4m} - 1 \right) m = q - m - 1 < q, \end{aligned}$$

as claimed. Note also that $n^* \geq 0$, since

$$\begin{aligned} \left\lfloor \frac{q-1}{4m} \right\rfloor - 1 &\geq \frac{q-1}{4m} - 2 \geq \frac{9(q-1)}{4q} - 2 = \\ &= \frac{1}{4} - \frac{9}{4q} \geq \frac{1}{4} - \frac{1}{4} = 0 \end{aligned}$$

Therefore the total number of points N_m with j in $\{\frac{q+1}{2} + m, \frac{q+1}{2} + m - 1\}$ satisfies

$$N_m \geq 2n^* + 1 = \left\lfloor \frac{q-1}{2m} \right\rfloor - 1$$

□

Proof of theorem 3. From the proof of theorem 1 it follows that the escaping orbit pass through the point $(x_0, (q-1)/2)$, where $x_0 = \frac{1}{4} + \frac{q-1}{2}$. There exist positive integers n_k for $k \in \{1, 2, \dots, \lfloor q/9 \rfloor\}$ such that

$$\begin{aligned} j_{n_k-2} &= \frac{q+1}{2} + k - 2, \quad j_{n_k-1} = \frac{q+1}{2} + k - 1 \\ j_{n_k} &= \frac{q+1}{2} + k \end{aligned}$$

since the orbit must pass through at least one point at each height.

By lemma 4, $\frac{q+1}{2} \leq r_{n_k} \leq \frac{q+1}{2} + k - 1$. Define

$$A_k = \left\{ (r_{n_k+m}, j_{n_k+m}) : m = 0, 1, \dots, \left\lfloor \frac{q-1}{2k} \right\rfloor - 2 \right\}$$

By lemma 5, $|A_k| = \left\lfloor \frac{q-1}{2k} \right\rfloor - 1$. Since

$$\begin{aligned} \frac{q}{2} < x_{n_k+m} < q, \quad y_{n_k+m} &= \frac{q+1}{2} + k \text{ for } m \text{ even and} \\ 0 < x_{n_k+m} < \frac{q}{2}, \quad y_{n_k+m} &= \frac{q+1}{2} + k - 1 \text{ for } m \text{ odd,} \end{aligned}$$

the A_k are disjoint. Therefore we have

$$\ell(q) \geq \sum_{k=1}^{\lfloor q/9 \rfloor} \left(\left\lfloor \frac{q-1}{2k} \right\rfloor - 1 \right) = O(q \log q)$$

as desired.

□

4 Periodic orbits.

We conclude our discussion with the numerical investigation of the distribution of the periodic orbits.

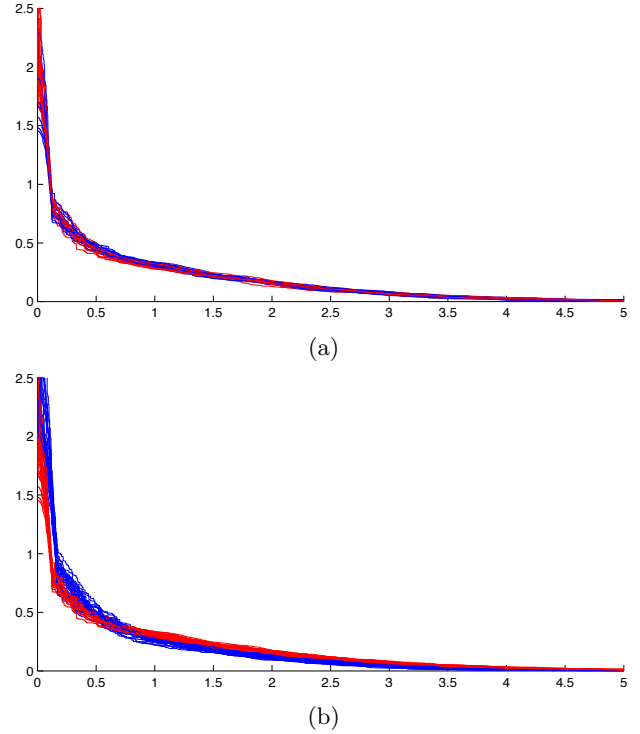


Figure 6: (a): Distribution of the preperiodic orbits for even $q = 950, \dots, 1000$. Case $q = 4k$ is drawn in red. Blue color corresponds to the case $q = 4k+2$. (b): Distribution of the preperiodic orbits for $q = 950, \dots, 1000$. Blue color corresponds to the case of odd q . Even q is drawn in red.

From theorem 1 it follows that the whole phase space of the system (3) is divided into the set of periodic orbits of various periods. If q is odd then there exists a unique orbit of enormously large period which swipes almost a half of the phase space. It turns out that all the other periods are distributed in the range of $O(q)$. For even q all the periods belong to this range. What is spectacular that we observe some similarity in the distribution of these periods for even and odd values of q . Collection of the periodic orbits represents

a partition of the number q^2 into the sum of the periods of the trajectories. We present here the Young diagrams for these partitions scaled by the factor of q in both directions. For the case of odd q we present the diagram corresponding to the partition of the set of bounded trajectories. It turns out that the Young diagrams constructed for the cases of even q and for the bounded part of the phase space for the odd q are similar (see Fig. 6b).

Conjecture 3. *Maximum length of the bounded trajectories for the transformation (3) has the magnitude $O(q)$.*

Looking at the portrait of the escaping trajectory (Fig. 3b) one can notice well-defined lacunae corresponding to the levels $j = q/(2n + 1)$. These lacunae represent the islands of stability around the corresponding periodic points for the transformation f .

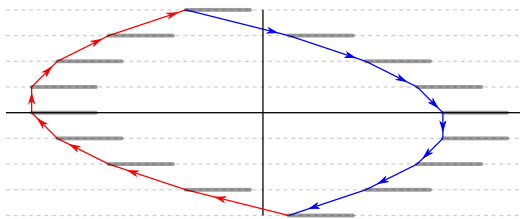


Figure 7: Lacuna near the level $j = 0$. Trajectories cover $O(q)$ distance in horizontal direction in $O(\sqrt{q})$ steps.

However, these islands do not exhaust the whole phase space since the every island consists of $O(q)$ bounded trajectories while every such trajectory has period of order $O(q^{1/2})$ (see Fig 7). Nevertheless we observe that

Conjecture 4. *Distributions of large periods of the periodic trajectories for even values of q coincide.*

On the other hand for small periods we have observed some differences. It turns out that for $q = 2 \pmod{4}$ number of periodic orbits of small periods does not depend on q while for $q = 4k$ there are exactly $(2k - 1)$ periodic orbits of period 4. Indeed one can easily check by the direct computation that trajectory of every point (r, k) , $r \in [2k, 3k - 1]$ is 4-periodic. Combined with the lemma 1 this observation provides $2k - 2$

points of period 4. Since the point $(0, 0)$ is clearly 4-periodic for any q , the total number of 4-periodic orbits equals $2k - 1$.

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